SPECTRAL MULTIPLICITY THEORY FOR A CLASS OF SINGULAR INTEGRAL OPERATORS(1)

BY

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This article is a continuation of the theory of singular integral operators initiated by the author and J. D. Pincus in a report [2], and continued subsequently in [1]. The operators which will be considered here have the form

$$(Lx)(\lambda) \equiv f(\lambda)x(\lambda) + \frac{1}{\pi i}\int_a^b \frac{(k(\lambda)k(\mu))^{1/2}}{\mu - \lambda}x(\mu)\,d\mu,$$

where $x(\lambda)$ is an arbitrary function of $L^2(a,b)$. The integral is to be interpreted as a Cauchy principal value, and $f(\lambda)$, $k(\lambda)$, are to be real valued, of class C^2 on $[a,b]$, such that the functions $f'(\lambda) \pm k'(\lambda)$ have only finitely many zeros, while $k(\lambda) > 0$ almost everywhere. Further, at each zero of $f'(\lambda) \pm k'(\lambda)$, we require the corresponding second derivative, $f''(k) \pm k''(\lambda)$, to be nonzero(2).

The contents of this paper have some points of contact with recent work of J. Schwartz [6], in which he analyzes the extent of the essential spectrum of general singular integral operators by using techniques of the theory of commutative Banach algebras. We consider only a special class of these operators, but our methods, which are fairly elementary in nature, allow us to obtain a complete spectral representation in this case. The important observation of Dr. Pincus that the hypothesis 6.1 of [1] is universally true motivated my search for a replacement of the lengthy computations of that article, and the main purpose of the discussion here is to give a concise argument leading to the main results which we now state(3).

THEOREM. *The operator L has a purely continuous spectrum which consists of the interval $[\alpha,\beta]$, $\alpha = \min\{f(\lambda) - k(\lambda)\}$, $\beta = \max\{f(\lambda) + k(\lambda)\}$. For almost all real ξ, the spectral multiplicity is the number D_ξ of connected components of the open set*

$$\{\lambda \in (a,b) \,|\, \{f(\lambda) - \xi - k(\lambda)\}\{f(\lambda) - \xi + k(\lambda)\}^{-1} < 0\}.$$

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(2) Actually, our techniques remain valid under a large number of hypotheses. For the sake of simplicity, we restrict the discussion to a simple case.

(3) *Added in proof.* A similar problem is treated in a paper by J. D. Pincus which appears in this issue of the Transactions.

Observe that the multiplicity is tied closely to the change of the argument of the function

$$G(\lambda, l) = \{f(\lambda) - l - k(\lambda)\}\{f(\lambda) - l + k(\lambda)\}^{-1}.$$

This is of interest in view of the fact that the change of argument of this function also enters the classical theory of singular integral equations in the discussion of the index [4].

1. **The fundamental solution.** As in [1], we rely mainly on the fundamental solution

$$E(l,z) = \exp\left\{\frac{1}{2\pi i}\int_a^b \log \frac{f(\mu) - l - k(\mu)}{f(\mu) - l + k(\mu)} \frac{d\mu}{\mu - z}\right\}$$

of the Hilbert problem

$$\{f(\lambda) - l + k(\lambda)\}X(\lambda + iO) - \{f(\lambda) - l - k(\lambda)\}X(\lambda - iO) = 0.$$

For the convenience of the reader, we list some formulae for $E(l,z)$ and the related functions,

$$H(l,\lambda) = E(l,\lambda + iO) - E(l,\lambda - iO), \qquad F(\xi,z) = E(\xi + iO,z) - E(\xi - iO,z).$$

$$\{f(\lambda) - l + k(\lambda)\}E(l,\lambda + iO) = \{f(\lambda) - l - k(\lambda)\}E(l,\lambda - iO), \tag{1.1}$$

$$\{f(\lambda) - l\}H(l,\lambda) = -k(\lambda)\{E(l,\lambda + iO) + E(l,\lambda - iO)\}, \tag{1.2}$$

$$\begin{aligned}(l_2 - l_1)&H(l_1,\lambda)\bar{H}(l_2,\lambda) \\ &= 2k(\lambda)\{E(l_1,\lambda + iO)\bar{E}(l_2,\lambda - iO) - E(l_1,\lambda - iO)\bar{E}(l_2,\lambda + iO)\}.\end{aligned} \tag{1.3}$$

With

$$\omega(\xi,z) = \frac{1}{\pi}\int_a^b \arg \frac{f(\mu) - \xi - iO - k(\mu)}{f(\mu) - \xi - iO + k(\mu)} \frac{d\mu}{\mu - z}, \qquad R(\xi,z) = \exp \omega(\xi,z),$$

we obtain

$$E(\xi + iO,z) = R(\xi,z)E(\xi - iO,z), \tag{1.4}$$

$$\frac{R(\xi,z) + 1}{R(\xi,z) - 1}F(\xi,z) = E(\xi + iO,z) + E(\xi - iO,z), \tag{1.5}$$

$$\begin{aligned}\frac{1}{2}&\left\{\frac{R(\xi,z_2) + 1}{R(\xi,z_2) - 1} - \frac{R(\xi,z_1) + 1}{R(\xi,z_1) - 1}\right\} F(\xi,z_1)\bar{F}(\xi,\bar{z}_2) \\ &= \{E(\xi + iO,z_1)\bar{E}(\xi - iO,\bar{z}_2) - E(\xi - iO,z_1)\bar{E}(\xi + iO,\bar{z}_2)\}.\end{aligned} \tag{1.6}$$

With our restrictive assumptions about $f(\lambda)$, $k(\lambda)$, the function $R(\xi,z)$ is, for each fixed real ξ, a rational function of z. For $\alpha < \xi < \beta$ ($\alpha = \min\{f(\lambda) - k(\lambda)\}$, $\beta = \max\{f(\lambda) + k(\lambda)\}$),

$$R(\xi,z) = g(\xi,z)/c(\xi,z),$$

where $c(\xi,z)$, $g(\xi,z)$ are relatively prime polynomials in z, with real coefficients and of the same degree, which we denote by D_ξ. If $r_1(\xi),\cdots,r_{D_\xi}(\xi)$ are the roots of $c(\xi,z)$ and $s_1(\xi),\cdots,s_{D_\xi}(\xi)$ the roots of $g(\xi,z)$, then the expression for $\omega(\xi,z)$ shows that we can number these roots, so that

$$a \leqq s_1(\xi) < r_1(\xi) < s_2(\xi) < r_2(\xi) < \cdots < s_{D_\xi}(\xi) < r_{D_\xi}(\xi) \leqq b. \tag{1.7}$$

We also note that both $c(\xi,z)$ and $g(\xi,z)$ have leading coefficient 1.

At this stage we apply the suggestion of Dr. Pincus and we replace the argument of §7 of [1] by using a partial fraction decomposition of the coefficient of $F(\xi,z)$ in formula (1.5). This will readily lead to the verification of hypothesis 6.1 of [1]. From (1.7), one easily deduces

LEMMA 1.1. *For each fixed $\xi \in (\alpha,\beta)$, the roots $\lambda_j(\xi)$ of the equation $R(\xi,z) = 1$ are all simple and contained in the interval (a,b). The number of roots is $D_\xi - 1$.*

In fact, the roots of $R(\xi,z) = 1$ coincide with the roots of the polynomial $Q(\xi,z) = c(\xi,z) - g(\xi,z)$ of degree $D_\xi - 1$. The location of these roots can be obtained by an examination of

$$\frac{g(\xi,z)}{c(\xi,z)} = \prod_{i=1}^{D_\xi} \frac{z - s_i(\xi)}{z - r_i(\xi)}$$

for real values of z. They can be numbered so that they satisfy

$$\begin{aligned} a \leqq s_1(\xi) < r_1(\xi) < \lambda_1(\xi) < s_2(\xi) < r_2(\xi) < \lambda_2(\xi) < \cdots &< \lambda_{D_\xi - 1}(\xi) \\ &< s_{D_\xi}(\xi) < r_{D_\xi}(\xi) \leqq b. \end{aligned} \tag{1.8}$$

Thus we have $R(\xi,\lambda_j(\xi)) = 1$, or $c(\xi,\lambda_j(\xi)) = g(\xi,\lambda_j(\xi))$. It is also not difficult to conclude that

$$\frac{dR}{dz}(\xi,\lambda_j(\xi)) = \frac{g'(\xi,\lambda_j(\xi)) - c'(\xi,\lambda_j(\xi))}{c(\xi,\lambda_j(\xi))}$$

is *negative*. We now set

$$a_j(\xi) = (-1)/R'(\xi,\lambda_j(\xi))$$

and

$$A(\xi) = -\frac{1}{\pi}\int_a^b \arg \frac{f(\mu) - \xi - iO - k(\mu)}{f(\mu) - \xi - iO + k(\mu)}\, d\mu.$$

By taking the pole at infinity into consideration, we find that the function

$$\frac{R(\xi,z)+1}{R(\xi,z)-1} - 2\left\{\frac{z}{A(\xi)} - \sum_j \frac{a_j(\xi)}{z - \lambda_j(\xi)}\right\}$$

is independent of z. Hence

$$\frac{1}{2}\left\{\frac{R(\xi,z_2)+1}{R(\xi,z_2)-1}-\frac{R(\xi,z_1)+1}{R(\xi,z_1)-1}\right\} = (z_2-z_1)\left\{\frac{1}{A(\xi)}+\sum_j \frac{a_j(\xi)}{(z_1-\lambda_j(\xi))(z_2-\lambda_j(\xi))}\right\}. \tag{1.9}$$

This formula contains the verification of hypothesis 6.1 of [1].

In order to continue, we shall subdivide the interval $[\alpha,\beta]$ as follows. We consider disjoint open subintervals I_ν, $\nu = 1,\cdots,N$ of $[\alpha,\beta]$, such that

$$[\alpha,\beta] = \bigcup_{\nu=1}^{N} \bar{I}_\nu, \tag{1.10}$$

$$\xi \in I_\nu \text{ and } \xi = f(\lambda) \pm k(\lambda) \rightarrow f'(\lambda) \pm k'(\lambda) \neq 0, \tag{1.11}$$

(1.12) the roots $s_j(\xi), r_j(\xi)$ can be defined as continuous functions for all $\xi \in I_\nu$, such that (1.7) holds almost everywhere in I_ν.

Our hypotheses on $f(\lambda)$, $k(\lambda)$ allow us to find intervals I_ν of this nature. An equality $s_j(\xi) = r_j(\xi)$ in (1.12) can then only occur if $f(s_j(\xi)) - \xi - k(s_j(\xi)) = f(s_j(\xi)) - \xi + k(s_j(\xi)) = 0$. Thus $k(s_j(\xi)) = 0$ and $f(s_j(\xi)) = \xi$ at such points. By our hypothesis, the set

$$N = \{\lambda \in [a,b] \mid k(\lambda) = 0\}$$

is of measure zero. Since the function $f(\lambda)$ is absolutely continuous, the set $f(N)$ also has measure zero. Since we had $f(s_j(\xi)) = \xi$, $k(s_j(\xi)) = 0$ for the exceptional points ξ of I_ν, this exceptional set lies in $f(N)$ and thus is of measure zero. Since the $s_j(\xi)$, $r_j(\xi)$ are roots of the equations $\xi = f(\lambda) \pm k(\lambda)$, they will even be of class C^2 on I_ν. Further, since the first and second derivatives of the functions $f(\lambda) \pm k(\lambda)$ never vanish simultaneously, these roots will satisfy a uniform Hölder condition with exponent $\frac{1}{2}$ on I_ν. This follows from the elementary fact, obtained by the use of Taylor's theorem, that if $h(\lambda)$ is of class C^2 on an interval, with $h'(\lambda) \neq 0$ and $|h''(\lambda)| \geqq M > 0$, then

$$|h(\lambda_1) - h(\lambda_2)| \geqq \frac{M}{2}(\lambda_1 - \lambda_2)^2$$

on that interval. With the aid of this discussion, together with the expression for $\omega(\xi,z)$, we immediately deduce

LEMMA 1.2. *For each fixed $z \notin [a,b]$, the function $\omega(\xi,z)$ is Hölder continuous for $-\infty < \xi < \infty$ and vanishes outside the interval (α,β).*

With $\xi \in I_\nu$, we shall denote the roots of $R(\xi,z) = 1$ by $\lambda_{j\,\nu}(\xi)$, $j = 2,\cdots,D_\xi$, $\nu = 1,\cdots,N$. For all $\xi \in I_\nu$ off our exceptional set of measure zero, D_ξ will be constant. We introduce the functions

$$a_{j,\nu}(\xi) = \begin{cases} (-1)/R'(\xi, \lambda_{j,\nu}(\xi)), & \xi \in I_\nu, \\ 0, & \xi \notin I_\nu, \end{cases}$$

$$a_{1,\nu}(\xi) = \begin{cases} 1/A(\xi), & \xi \in I_\nu, \\ 0, & \xi \notin I_\nu. \end{cases}$$

Thus, almost everywhere in I_ν, (1.9) becomes

$$\text{(1.9}') \quad \begin{aligned} &\frac{1}{2}\left\{\frac{R(\xi,z_2)+1}{R(\xi,z_2)-1} - \frac{R(\xi,z_1)+1}{R(\xi,z_1)-1}\right\} \\ &\qquad = (z_2 - z_1)\left\{a_{1,\nu}(\xi) + \sum_{j=2}^{D\xi} \frac{a_{j,\nu}(\xi)}{(z_1 - \lambda_{j,\nu}(\xi))(z_2 - \lambda_{j,\nu}(\xi))}\right\}. \end{aligned}$$

We now turn to

LEMMA 1.3. *For almost all $\xi \in I_\nu$, the functions $F(\xi, \lambda_{j,\nu}(\xi) \pm iO)$ vanish.*

Proof. By the formula in §9 of [1, p. 59],

$$F(\xi,z) = -\frac{Q(\xi,z)}{(c(\xi,z)\,g(\xi,z))^{1/2}} \exp\left\{\frac{1}{2\pi i}\int_a^b \log|G(\mu,\xi)|\frac{d\mu}{\mu - z}\right\},$$

where $Q(\xi,z) = c(\xi,z) - g(\xi,z)$, as before, and $G(\lambda, l)$ is the expression defined in the introduction. Hence, by the Plemelj formulae [4; 5],

$$\begin{aligned} &F(\xi, \lambda_{j,\nu}(\xi) + iO) \\ &\quad = -\frac{Q(\xi, \lambda_{j,\nu}(\xi))}{c(\xi, \lambda_{j,\nu}(\xi))}|G(\xi, \lambda_{j,\nu}(\xi))|^{1/2} \exp\left\{\frac{1}{2\pi i}\int_a^b \log|G(\mu,\xi)|\frac{d\mu}{\mu - \lambda_{j,\nu}(\xi)}\right\}. \end{aligned}$$

at all points $\xi \in I_\nu$ at which the function $\log|G(\xi, \lambda_{j,\nu}(\xi))|$ is defined. Now, off our exceptional set of measure zero, (1.8) holds, so that $G(\xi, \lambda_{j,\nu}(\xi)) \geqq 0$. Further, if for $\xi \in I_\nu$, $f(\lambda_{j,\nu}(\xi)) - \xi + k(\lambda_{j,\nu}(\xi)) = 0$, then, from (1.11) we see that $f(\lambda) - \xi + k(\lambda)$ changes sign at $\lambda_{j,\nu}(\xi)$. The same conclusion must hold for $f(\lambda) - \xi - k(\lambda)$, since the ratio $G(\xi, \lambda)$ must remain non-negative for λ near $\lambda_{j,\nu}(\xi)$. Hence, also $f(\lambda_{j,\nu}(\xi)) - \xi - k(\lambda_{j,\nu}(\xi)) = 0$, so that $k(\lambda_{j,\nu}(\xi)) = 0$. However, this can never happen off our exceptional set of measure zero. Using the fact that $Q(\xi, \lambda_{j,\nu}(\xi)) = 0$, we obtain our conclusion for $F(\xi, \lambda_{j,\nu}(\xi) + iO)$. The proof for $F(\xi, \lambda_{j,\nu}(\xi) - iO)$ is similar.

Finally, we mention that from properties of Cauchy integrals [4], one deduces that $E(l,z)$ has, for fixed $l \notin [\alpha, \beta]$, continuous boundary values as z approaches the real axis from above or below, at all points different from a and b. At a and b, the function is of order $O(1/(z-a)^\kappa)$, $O(1/(z-b)^\kappa)$, $\kappa < \frac{1}{2}$. Furthermore, for real values of l, both $E(l,z)$ and $E^{-1}(l,z)$ are even bounded. On the other hand, from Lemma 1.2 and formula (3.5) in [1], it follows that for fixed $z \notin [a,b]$, the function $E(l,z)$ has continuous boundary values at all points on the real axis.

2. **The spectral transformations.** We now define operators $R_{j,\nu}$ which are to act on functions of $L^2(a,b)$. We set

$$(R_{1,\nu}h)(\xi) = -\lim_{\delta\to 0+} \frac{1}{2\pi i}\int_a^b \left(\frac{a_{1,\nu}(\xi)}{2k(\lambda)}\right)^{1/2} \{\bar{H}(\xi + i\delta,\lambda) - \bar{H}(\xi - i\delta,\lambda)\}\, h(\lambda)d\lambda,$$

$$(R_{j,\nu}h)(\xi) = -\lim_{\varepsilon\to 0+}\left(\lim_{\delta\to 0+} \frac{1}{2\pi i}\int_a^b \left(\frac{a_{j,\nu}(\xi)}{2k(\lambda)}\right)^{1/2} \frac{\bar{H}(\xi + i\delta,\lambda) - \bar{H}(\xi - i\delta,\lambda)}{\lambda - \lambda_{j,\nu}(\xi) - i\varepsilon} h(\lambda)d\lambda\right), j \geqq 2.$$

With these definitions, we shall now prove

$$R_{1,\nu}\left\{\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - z)}\right\} = -\frac{(a_{1,\nu}(\xi))^{1/2}\,\bar{F}(\xi,\bar{z})\,E(l,z)}{\xi - l}, \tag{2.1}$$

$$R_{j,\nu}\left\{\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - z)}\right\} = -\frac{(a_{j,\nu}(\xi))^{1/2}\,\bar{F}(\xi,\bar{z})\,E(l,z)}{(z - \lambda_{j,\nu}(\xi))(\xi - l)}, \qquad j \geqq 2, \tag{2.2}$$

$$R_{1,\nu}\left\{\frac{(2k(\lambda))^{1/2}}{\lambda - z}\right\} = (a_{1,\nu}(\xi))^{1/2}\bar{F}(\xi,\bar{z}), \tag{2.3}$$

$$R_{j,\nu}\left\{\frac{(2k(\lambda))^{1/2}}{\lambda - z}\right\} = \frac{(a_{j,\nu}(\xi))^{1/2}\bar{F}(\xi,\bar{z})}{z - \lambda_{j,\nu}(\xi)}, \qquad j \geqq 2. \tag{2.4}$$

To prove (2.1), we consider the integral

$$\frac{1}{2\pi i}\oint \frac{\bar{E}(\bar{w},\zeta)\,E(l,z)}{\zeta - z}d\zeta = \bar{E}(\bar{w},\bar{z})\,E(l,z) - 1,$$

where the path of integration passes around a cut from a to b along the real axis, in a clockwise direction. In the limit, the left side of the equation becomes

$$\frac{1}{2\pi i}\int_a^b \{\bar{E}(\bar{w},\lambda - iO)\,E(l,\lambda + iO) - \bar{E}(\bar{w},\lambda + iO)\,E(l,\lambda - iO)\}\,(\lambda - z)^{-1}d\lambda$$

$$= \frac{w - l}{2\pi i}\int_a^b \frac{\bar{H}(\bar{w},\lambda)\,H(l,\lambda)}{2k(\lambda)(\lambda - z)}d\lambda,$$

where we have used (1.3). Thus we have

$$\frac{1}{2\pi i}\int_a^b \frac{\bar{H}(\bar{w},\lambda)}{(2k(\lambda))^{1/2}}\,\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - z)}d\lambda = \frac{1}{w - l}\,\{\bar{E}(\bar{w},\bar{z})\,E(l,z) - 1\}. \tag{2.5}$$

Consequently,

$$R_{1,\nu}\left\{\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - z)}\right\} = \frac{(a_{1,\nu}(\xi))^{1/2}E(l,z)}{\xi - l}\,\{\bar{E}(\xi - iO,\bar{z}) - \bar{E}(\xi + iO,\bar{z})\},$$

and formula (2.1) is established.

We turn next to (2.2). In this case we have

$$\frac{1}{2\pi i}\oint \frac{\bar{E}(\bar{w},\zeta)E(l,\zeta)}{(\zeta-z_1)(\zeta-z)}d\zeta = \frac{1}{z_1-z}\{E(l,z_1)\bar{E}(\bar{w},\bar{z}_1)-E(l,z)\bar{E}(\bar{w},\bar{z})\}$$

$$= \frac{w-l}{2\pi i}\int_a^b \frac{\bar{H}(\bar{w},\lambda)H(l,\lambda)}{2k(\lambda)(\lambda-z)(\lambda-z_1)}d\lambda$$

(from 1.3) so that

$$\frac{1}{2\pi i}\int_a^b \frac{\bar{H}(\bar{w},\lambda)H(l,\lambda)}{2k(\lambda)(\lambda-z)(\lambda-z_1)}d\lambda$$

(2.6)
$$= \frac{1}{(z_1-z)(w-l)}\{E(l,z_1)\bar{E}(\bar{w},\bar{z}_1)-E(l,z)\bar{E}(\bar{w},\bar{z}\}.$$

Consequently,

$$R_{j,\nu}\left\{\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda-z)}\right\}$$

$$= \frac{(a_{j,\nu}(\xi))^{1/2}}{(z-\lambda_{j,\nu}(\xi))(\xi-l)}\{E(l,z)(\bar{E}(\xi-iO,\bar{z})-\bar{E}(\xi+iO,\bar{z}))$$
$$+E(l,\lambda_{j,\nu}(\xi)+iO)\bar{F}(\xi,\lambda_{j,\nu}(\xi)-iO)\}.$$

Since $\bar{F}(\xi,\lambda_{j,\nu}(\xi)-iO)=0$ almost everywhere on I_ν, we obtain the required result.

To establish (2.3), we consider

$$\frac{1}{2\pi i}\oint \frac{E(l,\zeta)}{\zeta-z}d\zeta = E(l,z)-1 = \frac{1}{2\pi i}\int_a^b \frac{H(l,\lambda)}{\lambda-z}\,d\lambda.$$

Thus

(2.7)
$$-\frac{1}{2\pi i}\int_a^b \frac{\bar{H}(l,\lambda)}{\lambda-z}d\lambda = \bar{E}(l,\bar{z})-1$$

and

$$R_{1,\nu}\left\{\frac{(2k(\lambda))^{1/2}}{\lambda-z}\right\} = (a_{1,\nu}(\xi))^{1/2}\bar{F}(\xi,\bar{z}).$$

Finally, to prove (2.4), we consider

$$\frac{1}{2\pi i}\oint \frac{E(l,\zeta)}{(\zeta-z)(\zeta-z_1)}d\zeta = \frac{1}{z-z_1}\{E(l,z)-E(l,z_1)\}$$

$$= \frac{1}{2\pi i}\int_a^b \frac{H(l,\lambda)}{(\lambda-z)(\lambda-z_1)}d\lambda.$$

Thus

(2.8)
$$-\frac{1}{2\pi i}\int_a^b \frac{\bar{H}(l,\lambda)}{(\lambda-z)(\lambda-z_1)}d\lambda = \frac{1}{z-z_1}\{\bar{E}(l,\bar{z})-\bar{E}(l,\bar{z}_1)\},$$

and

$$R_{j,\nu}\left\{\frac{(2k(\lambda))^{1/2}}{\lambda - z}\right\} = \frac{(a_{j,\nu}(\xi))^{1/2}}{z-\lambda_{j,\nu}(\xi)}\{\bar{F}(\xi,\bar{z}) - \bar{F}(\xi,\lambda_{j,\nu}(\xi) - iO)\}.$$

Since $\bar{F}(\xi,\lambda_{j,\nu}(\xi) - iO) = 0$, we have the required result.

3. **The isometry.** In this section, we shall consider the linear manifold in $L^2(a,b)$ spanned by the functions

$$\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - z)}. \tag{3.1}$$

We shall show that for any functions $h(\lambda)$, $g(\lambda)$ of this manifold,

$$\int_a^b h(\lambda)\bar{g}(\lambda)d\lambda = \sum_{j,\nu}\int_\alpha^\beta (R_{j,\nu}h)(\overline{R_{j,\nu}g})d\xi. \tag{3.2}$$

To do this, we consider

$$\frac{1}{2\pi i}\oint \frac{\bar{E}(\bar{w},\bar{z}_1)E(w,z_2)}{(w-l_1)(w-l_2)}dw = \frac{\bar{E}(l_1,\bar{z}_1)E(l_1,z_2)}{l_1-l_2} + \frac{\bar{E}(l_2,\bar{z}_1)E(l_2,z_2)}{l_2-l_1},$$

where the path of integration passes around a cut from α to β along the real axis, in a clockwise direction. In the limit, the left side becomes

$$\frac{1}{2\pi i}\int_\alpha^\beta \{\bar{E}(\xi - iO,\bar{z}_1)E(\xi+iO,z_2) - \bar{E}(\xi+iO,\bar{z}_1)E(\xi-iO,z_2)\}(\xi-l_1)^{-1}(\xi-l_2)^{-1}d\xi$$

$$= \frac{1}{2\pi i}\int_\alpha^\beta \frac{1}{2}\left\{\frac{R(\xi,z_1)+1}{R(\xi,z_1)-1} - \frac{R(\xi,z_2)+1}{R(\xi,z_2)-1}\right\}\frac{\bar{F}(\xi,\bar{z}_1)F(\xi,z_2)}{(\xi-l_1)(\xi-l_2)}d\xi,$$

where we have used (1.6). Using (1.9′), we obtain

$$\begin{aligned}\frac{z_1-z_2}{2\pi i}\int_\alpha^\beta\left(\sum_\nu a_{1,\nu}(\xi) + \sum_{j\geqq 2,\nu}\frac{a_{j,\nu}(\xi)}{(z_1-\lambda_{j,\nu}(\xi))(z_2-\lambda_{j,\nu}(\xi))}\right)\frac{\bar{F}(\xi,\bar{z}_1)F(\xi,z_2)}{(\xi-l_1)(\xi-l_2)}d\xi \\ = \frac{\bar{E}(l_1,\bar{z}_1)E(l_1,z_2)}{l_1-l_2} + \frac{\bar{E}(l_2,\bar{z}_1)E(l_2,z_2)}{l_2-l_1}.\end{aligned} \tag{3.3}$$

We set

$$P(\xi,z_1,z_2) = \sum_\nu a_{1,\nu}(\xi) + \sum_{j\geqq 2,\nu}\frac{a_{j,\nu}(\xi)}{(z_1-\lambda_{j,\nu}(\xi))(z_2-\lambda_{j,\nu}(\xi))}.$$

From (3.3), we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_\alpha^\beta P(\xi,z_1,z_2)\frac{E(l_1,z_1)\bar{F}(\xi,\bar{z}_1)\bar{E}(l_2,\bar{z}_2)F(\xi,z_2)}{(\xi-l_1)(\xi-l_2)}d\xi \\ = \frac{E(l_1,z_1)\bar{E}(l_2,\bar{z}_2)}{z_1-z_2}\left\{\frac{\bar{E}(l_1,\bar{z}_1)E(l_1,z_2)}{l_1-l_2} + \frac{\bar{E}(l_2,\bar{z}_1)E(l_2,z_2)}{l_2-l_1}\right\}.\end{aligned} \tag{3.4}$$

Now we observe that

$$(3.5) \qquad \bar{E}(l,z) = 1/E(l,z).$$

Thus the right side of (3.4) finally becomes

$$(3.6) \qquad \frac{1}{(z_1 - z_2)\,(l_1 - l_2)}\{E(l_1,z_2)\bar{E}(l_2,\bar{z}_2) - E(l_1,z_1)\bar{E}(l_2,\bar{z}_1)\}.$$

However, if we compare (3.6) with formula (2.6), we obtain

$$(3.7) \int_a^b \frac{\bar{H}(l_2,\lambda)H(l_1,\lambda)}{2k(\lambda)(\lambda-z_1)(\lambda-z_2)}d\lambda = \int_\alpha^\beta P(\xi,z_1,z_2)\frac{E(l_1,z_1)\bar{F}(\xi,z_1)\bar{E}(l_2,\bar{z}_2)F(\xi,z_2)d\xi}{(\xi-l_1)(\xi-l_2)}.$$

Formula (3.7), together with (2.1), (2.2) and the expression for $P(\xi,z_1,z_2)$ proves the assertion (3.2).

4. **The bounded operators P and Q.** Since the linear manifold spanned by the elements $\{1/(\lambda - z)\}$ is dense in $L^2(a,b)$, it is not difficult to conclude from the properties of $E(l,z)$, stated at the end of §1, and from formulae (1.1), (1.2), that the linear manifold spanned by the functions (3.1) is dense in $L^2(a,b)$. From (3.2), we conclude that all of the operators $R_{j,\nu}$ are bounded, with norm $\leqq 1$ on this manifold. Consequently, each $R_{j,\nu}$ possesses a unique bounded extension $P_{j,\nu}$, defined on all of $L^2(a,b)$, with norm $\leqq 1$. Further, from (3.2) we obtain the relation

$$(4.1) \qquad (x,y) = \sum_{j\,\nu} (P_{j,\nu}x, P_{j,\nu}y)$$

for all $x, y \in L^2(a,b)$.

Observing that

$$(4.2) \qquad |(x,P_{j,\nu}y)| \leqq \|x\|\;\|P_{j,\nu}y\| \leqq \|x\|\;\|y\|,$$

for $x \in L^2(I_\nu)$, $y \in L^2(a,b)$, we see that for each fixed $x \in L^2(I_\nu)$, $(x, P_{j,\nu}y)$ is a bounded linear functional on $L^2(a,b)$. By the Riesz representation theorem, this functional may be represented in the form

$$(4.3) \qquad (x,P_{j,\nu}y) = (Q_{j,\nu}x, y),$$

where $Q_{j,\nu}x \in L^2(a,b)$. The norm of this functional is $\|Q_{j,\nu}x\|$. Since (4.2) provides the bound $\|x\|$ for this functional, we get

$$(4.4) \qquad \|Q_{j,\nu}x\| \leqq \|x\|.$$

In this way we define the bounded linear operators $Q_{j,\nu}$ on $L^2(I_\nu)$. They have norm $\leqq 1$. In view of (4.3) and (4.1), we have

$$(4.5) \qquad (x,y) = \sum_{j,\nu}(Q_{j,\nu}P_{j,\nu}x, y) = \Big(\sum_{j,\nu} Q_{j,\nu}P_{j,\nu}x, y\Big)$$

for all $x, y \in L^2(a,b)$. This relation immediately implies

$$1 = \sum_{j,\nu} Q_{j,\nu} P_{j,\nu}. \tag{4.6}$$

Observe, further, that (4.3), (2.1), (2.2) imply the relations

$$\int_a^b \frac{\bar{H}(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - \bar{z})}(Q_{1,\nu}g)(\lambda)d\lambda = - \int_\alpha^\beta \frac{(a_{1,\nu}(\xi))^{1/2}F(\xi,\bar{z})\bar{E}(l,z)}{\xi - l} g(\xi)d\xi, \tag{4.7}$$

$$\int_a^b \frac{\bar{H}(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - \bar{z})}(Q_{j,\nu}g)(\lambda)d\lambda = - \int_\alpha^\beta \frac{(a_{j,\nu}(\xi))^{1/2}F(\xi,\bar{z})\bar{E}(l,z)}{(\bar{z} - \lambda_{j,\nu}(\xi))(\xi - l)} g(\xi)d\xi. \tag{4.8}$$

5. **The orthonormality relations.** In this section, we shall prove that

$$R_{m,\mu}Q_{n,\nu} = \delta_{mn,\mu\nu}\cdot 1, \tag{5.1}$$

where $\delta_{mn,\mu\nu} = 0$ whenever either $m \neq n$ or $\mu \neq \nu$, while $\delta_{mm,\mu\mu} = 1$. We first discuss the case $m = 1$. For this, we take the identity (4.7), multiply by $-\bar{z}$, and let $z \to \infty$. Then one can readily see that

$$\lim_{z\to\infty} -\bar{z}F(\xi,\bar{z}) = -A(\xi), \tag{5.2}$$

$$\lim_{z\to\infty} \bar{E}(l,z) = 1. \tag{5.3}$$

Hence we get

$$\int_a^b \frac{\bar{H}(l,\lambda)}{(2k(\lambda))^{1/2}}(Q_{1,\nu}g)(\lambda)d\lambda = \int_{I_\nu} \frac{A((\xi))^{1/2}}{\xi - l} g(\xi)d\xi. \tag{5.4}$$

If we apply the Plemelj formulae, we get

$$\begin{aligned} - \lim_{\delta\to 0+} \frac{1}{2\pi i}\int_a^b \frac{\bar{H}(\xi + i\delta,\lambda) - \bar{H}(\xi - i\delta,\lambda)}{(2k(\lambda))^{1/2}}(Q_{1,\nu}g)(\lambda)d\lambda \\ = \chi_{I_\nu}(\xi)(A(\xi))^{1/2}g(\xi), \end{aligned} \tag{5.5}$$

where χ_{I_ν} is the characteristic function of I_ν. Thus

$$(R_{1,\mu}Q_{1,\nu}g)(\xi) = \chi_{I\nu}(\xi)(a_{1,\mu}(\xi))^{1/2}(A(\xi))^{1/2}g(\xi). \tag{5.6}$$

The right side is zero for $\mu \neq \nu$, because the supports of χ_{I_ν} and $a_{1,\mu}$ then lie in different intervals. However, for $\mu = \nu$, $a_{1,\nu}(\xi) = 1/A(\xi)$, $\xi \in I_\nu$, so that $(R_{1,\nu}Q_{1,\nu}g)(\xi) = g(\xi)$. This proves (5.1) for the case $m = n = 1$.

We turn now to the cases where $m = 1$, $n \neq 1$. Here we multiply the identity (4.8) by $-\bar{z}$. Because

$$\lim_{z\to\infty} F(\xi,\bar{z}) = 0, \tag{5.7}$$

we obtain

$$\int_a^b \frac{\bar{H}(l,\lambda)}{(2k(\lambda))^{1/2}}(Q_{j,\nu}g)(\lambda)d\lambda = 0. \tag{5.8}$$

Hence, we must conclude that $(R_{1,\mu}Q_{j,\nu}g)(\xi) = 0, j \neq 1$.

Next, we treat the cases where $m, n \neq 1$. Here we apply the Plemelj formulae to (4.8), to obtain

$$(5.9)\quad \begin{aligned} &-\lim_{\delta\to 0+}\frac{1}{2\pi i}\int_a^b \frac{\bar{H}(\xi+i\delta,\lambda)-\bar{H}(\xi-i\delta,\lambda)}{(2k(\lambda))^{1/2}(\lambda-\bar{z})}(Q_{n,\nu}g)(\lambda)d\lambda \\ &= -\frac{1}{2}\frac{(a_{n,\nu}(\xi))^{1/2}}{\bar{z}-\lambda_{n,\nu}(\xi)}F(\xi,\bar{z})g(\xi)\{\bar{E}(\xi+iO,z)+\bar{E}(\xi-iO,z)\} \\ &\quad -\frac{\bar{F}(\xi,z)}{2\pi i}\int_\alpha^\beta \frac{(a_{n,\nu}(\eta))^{1/2}F(\eta,\bar{z})}{(\bar{z}-\lambda_{n,\nu}(\eta))(\eta-\xi)}g(\eta)d\eta. \end{aligned}$$

However, using (1.5), we can write the right side in the form

$$-\frac{1}{2}\frac{(a_{n,\nu}(\xi))^{1/2}}{\bar{z}-\lambda_{n,\nu}(\xi)}\frac{\bar{R}(\xi,z)+1}{\bar{R}(\xi,z)-1}F(\xi,\bar{z})\bar{F}(\xi,z)g(\xi)$$
$$-\frac{\bar{F}(\xi,z)}{2\pi i}\int_\alpha^\beta \frac{(a_{n,\nu}(\eta))^{1/2}F(\eta,\bar{z})}{(\bar{z}-\lambda_{n,\nu}(\eta))(\eta-\xi)}g(\eta)d\eta.$$

Again, by the formula in §9 of [1],

$$(5.10)\quad F(\xi,\bar{z})\bar{F}(\xi,z)=\frac{(Q(\xi,\bar{z}))^2}{c(\xi,\bar{z})g(\xi,\bar{z})}.$$

With $\xi\in I_\mu$, we now let $z\to\lambda_{m,\mu}(\xi)-iO$ in (5.9). Since $F(\xi,\lambda_{m,\mu}(\xi)-iO)=0$, we obtain

$$(5.11)\quad \begin{aligned} &-\lim_{\varepsilon\to 0+}\lim_{\delta\to 0+}\frac{1}{2\pi i}\int_a^b \frac{\bar{H}(\xi+i\delta,\lambda)-\bar{H}(\xi-i\delta,\lambda)}{(2k(\lambda))^{1/2}(\lambda-\lambda_{m,\mu}(\xi)-i\varepsilon)}(Q_{n,\nu}g)(\lambda)\,d\lambda \\ &= \lim_{z\to\lambda_{m,\mu}(\xi)}\left\{\frac{\bar{z}}{A(\xi)}-\sum_{j\geqq 2}\frac{a_{j,\mu}(\xi)}{\bar{z}-\lambda_{j,\mu}(\xi)}+c_\mu(\xi)\right\} \\ &\quad\times\frac{(Q(\xi,\bar{z}))^2(a_{n,\nu}(\xi))^{1/2}g(\xi)}{c(\xi,\bar{z})g(\xi,\bar{z})(\bar{z}-\lambda_{n,\nu}(\xi))},\qquad \xi\in I_\mu, \end{aligned}$$

where we have made use of the expression which precedes formula (1.9). We can now compute $(R_{m,\mu}Q_{n,\nu}g)(\xi)$, by multiplying both sides of (5.11) by $(a_{m,\mu}(\xi))^{1/2}$. Of course, if $\mu\neq\nu$, then $(R_{m,\mu}Q_{n,\nu}g)(\xi)$ will be zero since the support of $a_{n,\nu}(\xi)$ is contained in I_ν. Thus we have established (5.1) for $\mu\neq\nu$. Since $Q(\xi,\lambda_{m,\nu}(\xi))=0$, one can also easily establish that $(R_{m,\nu}Q_{n,\nu}g)(\xi)=0$ for $m\neq n$. Finally for $m=n$, $\mu=\nu$, we get

$$(5.12)\quad (R_{n,\nu}Q_{n,\nu}g)(\xi)=\frac{(a_{n,\nu}(\xi))^2(Q'(\xi,\lambda_{n,\nu}(\xi)))^2g(\xi)}{(c(\xi,\lambda_{n,\nu}(\xi)))^2}.$$

However,

$$a_{n,\nu}(\xi) = \frac{c(\xi,\lambda_{n,\nu}(\xi))}{c'(\xi,\lambda_{n,\nu}(\xi)) - g'(\xi,\lambda_{n,\nu}(\xi))},$$

$$Q'(\xi,\lambda_{n,\nu}(\xi)) = c'(\xi,\lambda_{n,\nu}(\xi)) - g'(\xi,\lambda_{n,\nu}(\xi)).$$

Hence $(R_{n,\nu}Q_{n,\nu}g)(\xi) = g(\xi)$, and formula (5.1) is established for this case.

In order to discuss the case $n = 1$, $m \neq 1$, we apply the Plemelj formulae to (4.7) to obtain

$$\begin{aligned} -\lim_{\delta\to 0+} \frac{1}{2\pi i}\int_a^b \frac{\bar{H}(\xi+i\delta,\lambda)-\bar{H}(\xi-i\delta,\lambda)}{(2k(\lambda))^{1/2}(\lambda-\bar{z})}(Q_{1,\nu}g)(\lambda)d\lambda \\ = -(a_{1,\nu}(\xi))^{1/2}F(\xi,\bar{z})\bar{E}(l,z)g(\xi). \end{aligned} \tag{5.13}$$

Now we let $z \to \lambda_{m,\mu}(\xi) - iO$. However, since $F(\xi,\lambda_{m,\mu}(\xi)+iO) = 0$, we must have $(R_{m,\mu}Q_{1,\nu}g)(\xi) = 0$. This completes the proof of (5.1).

6. **The identity of the operators R and P.** In § 5, we established that the operator $R_{j,\nu}$ which we defined in §2, can be applied to any element in the range of an operator $Q_{m,\mu}$. However, in (4.6), we saw that every $x \in L^2(a,b)$ may be written in the form $x = \sum_{j,\nu} Q_{j,\nu}P_{j,\nu}x$.

With this equation and the orthonormality relations, we immediately deduce

THEOREM 6.1. *The operators $R_{j,\nu}$ are defined for all elements $x \in L^2(a,b)$ and $R_{j,\nu} = P_{j\,\nu}$.*

COROLLARY. *The operators $Q_{j,\nu} \circ P_{j,\nu}$ are orthogonal projections defined on $L^2(a,b)$, with mutually orthogonal ranges.*

7. **The spectral representation of the operator L.** In order to see that we have a spectral decomposition of our operator L, we shall first show that

$$L\left\{\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda-z)}\right\} = \frac{lH(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda-z)} - \frac{(2k(\lambda))^{1/2}E(l,z)}{\lambda-z}. \tag{7.1}$$

We begin by considering the function

$$G(\zeta) = \frac{1}{2\pi i}\int_a^b \frac{H(l,\mu)}{\mu-z}\,\frac{d\mu}{\mu-\zeta}. \tag{7.2}$$

Then we have

$$G(\lambda+iO) - G(\lambda-iO) = \frac{H(l,\lambda)}{\lambda-z}, \qquad \lambda \in (a,b), \tag{7.3}$$

$$\begin{aligned} \frac{1}{2\pi i}\int_a^b \frac{H(l,\mu)}{(\mu-z)(\mu-\zeta)}d\mu = \frac{1}{2\pi i}\oint \frac{E(l,w)}{(w-z)(w-\zeta)}dw \\ = \frac{E(l,z)-E(l,\zeta)}{z-\zeta}. \end{aligned} \tag{7.4}$$

This implies

$$
\begin{aligned}
G(\lambda + i0) + G(\lambda - i0) &= \frac{E(l,\lambda + i0) + E(l,\lambda - i0)}{\lambda - z} - 2\,\frac{E(l,z)}{\lambda - z} \\
&= \frac{1}{\pi i}\int_a^b \frac{H(l,\mu)}{\mu - z}\,\frac{d\mu}{\mu - \lambda}.
\end{aligned} \tag{7.5}
$$

Hence

$$
\begin{aligned}
f(\lambda)&\left\{\frac{H(l,\lambda)}{(\lambda - z)}\right\} + \frac{k(\lambda)}{\pi i}\int_a^b \frac{H(l,\mu)}{\mu - z}\,\frac{d\mu}{\mu - \lambda} \\
&= \frac{f(\lambda)}{\lambda - z}H(l,\lambda) + \frac{k(\lambda)}{\lambda - z}\{E(l,\lambda + i0) + E(l,\lambda - i0)\} - \frac{2k(\lambda)E(l,z)}{\lambda - z} \\
&= \frac{lH(l,\lambda)}{\lambda - z} - \frac{2k(\lambda)E(l,z)}{\lambda - z},
\end{aligned}
$$

where we have used (1.2). (7.1) now follows from this last expression.

Next, we shall verify the formulae

$$
R_{1,\nu}L\left\{\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - z)}\right\} = -\frac{\xi(a_{1,\nu}(\xi))^{1/2}\bar{F}(\xi,\bar{z})E(l,z)}{\xi - l}, \tag{7.6}
$$

$$
R_{j,\nu}L\left\{\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - z)}\right\} = -\frac{\xi(a_{j,\nu}(\xi))^{1/2}\bar{F}(\xi,\bar{z})E(l,z)}{(z - \lambda_{j,\nu}(\xi))(\xi - l)}, \qquad j \geqq 2. \tag{7.7}
$$

In fact, using (7.1), (2.1), and (2.3), we get

$$
\begin{aligned}
R_{1,\nu}L&\left\{\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - z)}\right\} \\
&= -\frac{l(a_{1,\nu}(\xi))^{1/2}\bar{F}(\xi,\bar{z})E(l,z)}{\xi - l} - (a_{1,\nu}(\xi))^{1/2}\bar{F}(\xi,\bar{z})E(l,z),
\end{aligned}
$$

which yields (7.6). Using (7.1), (2.2), (2.4), we get

$$
\begin{aligned}
R_{j,\nu}L&\left\{\frac{H(l,\lambda)}{(2k(\lambda))^{1/2}(\lambda - z)}\right\} \\
&= -\frac{l(a_{j,\nu}(\xi))^{1/2}\bar{F}(\xi,\bar{z})E(l,z)}{(z - \lambda_{j,\nu}(\xi))(\xi - l)} - \frac{(a_{j,\nu}(\xi))^{1/2}\bar{F}(\xi,\bar{z})E(l,z)}{z - \lambda_{j,\nu}(\xi)}, \quad j \geqq 2,
\end{aligned}
$$

which proves (7.7).

Since both $R_{j,\nu}$ and L are bounded operators (cf. [3]), we now have

THEOREM 7.1. *For each $h \in L^2(a,b)$, $(R_{j,\nu}Lh)(\xi) = \xi(R_{j,\nu}h)(\xi)$.*

COROLLARY. *The projections $Q_{j,\nu}P_{j,\nu}$ commute with the operator L.*

In fact, if $h(\lambda) = (Q_{j,\nu}g)(\lambda)$, then

$$(P_{m,\mu}Lh)(\xi) = \xi(P_{m,\mu}h)(\xi) = \xi(P_{m,\mu}Q_{j,\nu}g)(\xi)$$
$$= \xi(\delta_{mj,\mu\nu} \cdot 1 \cdot g)(\xi).$$

On the other hand,

$$Lh = \sum_{m,\mu} Q_{m,\mu}P_{m,\mu}Lh.$$

It follows that $Lh = Q_{j,\nu}P_{j,\nu}h$, so that the range of $Q_{j,\nu}$ is invariant under L.

From these results and the fact that $A(\xi) > 0$ almost everywhere on $[\alpha, \beta]$ (cf. [1, §4]), the main theorem stated in the introduction follows immediately.

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